

# On Weil numbers in cyclotomic fields\*

Bruno Anglès, Tatiana Beliaeva

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Let  $p$  be an odd prime number. It was noticed by Iwasawa that the  $p$ -adic behavior of Jacobi sums in  $\mathbb{Q}(\zeta_p)$  is linked to Vandiver's Conjecture (see [Iw]). This result has been generalized by various authors for the cyclotomic  $\mathbb{Z}_p$ -extensions of abelian fields (see for example [HI], [I1], [B]). In this paper we consider the module of Weil numbers (see §2) for the cyclotomic  $\mathbb{Z}_p$ -extension of  $\mathbb{Q}(\zeta_p)$ , and we get some results quite similar to those for Jacobi sums. In particular we establish a connection between the  $p$ -adic behavior of Weil numbers and a weak form of Greenberg Conjecture (see [N2], [BN])

## 1 Notations

Let  $p$  be a fixed odd prime number. For any  $n \in \mathbb{N}$  we denote by  $k_n$  the  $p^{n+1}$ -th cyclotomic field  $\mathbb{Q}(\mu_{p^{n+1}})$ , where  $\mu_{p^{n+1}}$  is the group of  $p^{n+1}$ -th roots of unity. We note  $\Delta = \text{Gal}(k_0/\mathbb{Q})$ ,  $\Gamma_n = \text{Gal}(k_n/k_0)$  and  $G_n = \text{Gal}(k_n/\mathbb{Q})$ , so  $G_n \simeq \Delta \times \Gamma_n$ . Let  $\zeta_p \in \mu_p \setminus \{1\}$  and take for any  $n \in \mathbb{N}$   $\zeta_{p^{n+1}} \in \mu_{p^{n+1}}$  such that  $\forall n \geq 1$   $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ . We note  $\pi_n = 1 - \zeta_p^{n+1}$

We shall also use the following more or less standard notations:

$k_{n,p}$  the  $p$ -completion of  $k_n$ ;

$\mathcal{U}_n = 1 + \pi_n \mathbb{Z}_p[\zeta_{p^{n+1}}]$  principal units in  $k_{n,p}$ ;

$\Gamma = \varprojlim \Gamma_n \simeq \mathbb{Z}_p$ ,  $\gamma_0$  its topological generator, where  $\forall \varepsilon \in \mu_{p^\infty}$ ,  $\gamma_0(\varepsilon) = \varepsilon^{1+p}$ ;

$\Lambda = \mathbb{Z}_p[[\Gamma]]$  the Iwasawa algebra of the profinite group  $\Gamma$ ,  $\Lambda \simeq \mathbb{Z}_p[[T]]$  by sending  $\gamma_0 - 1$  to  $T$  ([W, Theorem 7.1]);

$A_n$  is the Sylow  $p$ -subgroup of  $Cl(k_n)$ , where  $Cl(k_n)$  is the ideal class group of  $k_n$ ;

$X = \varprojlim A_n$  be the projective limit of  $A_n$  for the norm maps;

$I_n$  the group of prime-to- $p$  ideals of  $k_n$ ;

$k_\infty = \bigcup_{n \in \mathbb{N}} k_n$ ,  $\text{Gal}(k_\infty/k_0) = \Gamma$ ;

$L_n/k_n$  the maximal abelian unramified  $p$ -extension of  $k_n$ ;  $\text{Gal}(L_n/k_n) \simeq A_n$  by class field theory;

$M_n/k_n$  the maximal abelian  $p$ -extension of  $k_n$  unramified outside of  $p$ ;

$\mathfrak{X}_n = \text{Gal}(M_n/k_n)$ ;

$L_\infty = \bigcup L_n$ ;  $X \simeq \text{Gal}(L_\infty/K_\infty)$ ;

$M_\infty = \bigcup M_n$ ;

$\mathfrak{X}_\infty = \text{Gal}(M_\infty/k_\infty)$ .

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Let  $\psi$  be a fixed odd character of  $\Delta$ , different from Teichmüller character  $\omega$ . We note  $e_\psi$  the associated idempotent defined by

$$e_\psi = \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \psi(\delta) \delta^{-1} \in \mathbb{Z}_p[\Delta].$$

Let  $\mathcal{M} \in \Lambda$  be the distinguished polynomial of smallest degree such that  $\mathcal{M}(T)e_\psi X = \{0\}$ . We call it *the minimal polynomial of  $e_\psi X$* . It is well known to be prime to  $\omega_n = (T+1)^{p^n} - 1$  for any  $n$  (cf. [W, §13.6, Theorem 7.10, Theorem 5.11 and Theorem 4.17]).

## 2 Weil numbers and Jacobi sums.

Fix an  $n$  for a moment.

**Definition 1** We call Weil module of  $k_n$  the module  $\mathcal{W}_n$  defined by

$$\mathcal{W}_n = \left\{ f \in \text{Hom}_{\mathbb{Z}[G_n]}(I_n, k_n^*) \mid \exists \beta(f) \in \mathbb{Z}[G_n] \text{ such that } \forall \mathfrak{a} \in I_n \right. \\ \left. \mathfrak{a} = (\alpha) \Rightarrow f(\mathfrak{a}) \equiv \alpha^{\beta(f)} \pmod{\mu_{2p^{n+1}}} \right\} \quad (1)$$

Observe that  $\forall f \in \mathcal{W}_n, f(I_n) \subset \mu_{2p^{n+1}} \mathcal{U}_n$ .

**Definition 2** So we define the module of Weil numbers  $W_n$

$$W_n = \{ f(\mathfrak{a}) \mid f \in \mathcal{W}_n, \mathfrak{a} \in I_n \}.$$

Observe that  $W_n$  is a submodule of  $\mu_{2p^{n+1}} \mathcal{U}_n$ .

Let  $k_n^+$  be the maximal totally real subfield of  $k_n$  and let  $G_n^+$  stay for  $\text{Gal}(k_n^+/\mathbb{Q})$ . Let  $N_n$  be the norm element in  $\mathbb{Z}[G_n]$ . Let  $N_n^+ \in \mathbb{Z}[G_n]$  be such that its image by the restriction map  $\mathbb{Z}[G_n] \longrightarrow \mathbb{Z}[G_n^+]$  is  $\sum_{\sigma \in G_n^+} \sigma$ .

**Lemma 1** Let  $f \in \mathcal{W}_n$ . Then  $\beta(f) \in \mathbb{Z}[G_n]$  is unique and

$$\beta(f) \in N_n^+ \mathbb{Z} + \mathbb{Z}[G_n]^-.$$

**Proof:** Let  $f$  be in  $\mathcal{W}_n$  and suppose  $\beta(f)$  and  $\beta'(f)$  verify the required condition.

Let  $\mathfrak{p}$  be a split prime ideal in  $I_n$ . Let  $m \geq 1$  be such that  $\mathfrak{p}^m = \alpha \mathcal{O}_{k_n}$ . Then

$$f(\mathfrak{p}^m) \equiv \alpha^{\beta(f)} \equiv \alpha^{\beta'(f)} \pmod{\mu_{2p^{n+1}}}.$$

Thus  $\mathfrak{p}^{m\beta(f)} = \mathfrak{p}^{m\beta(f')}$ , that implies  $\mathfrak{p}^{m(\beta(f)-\beta(f'))} = \mathcal{O}_{k_n}$ , so  $\beta(f) = \beta'(f)$ . Furthermore:

$$\beta(f) \in \text{Ann}_{\mathbb{Z}[G_n]}(\mathcal{O}_{k_n}^* / \mu_{2p^{n+1}}) = N_n^+ \mathbb{Z} + \mathbb{Z}[G_n]^-. \quad \square$$

**Proposition 1** *The map  $\beta : \mathcal{W}_n \longrightarrow N_n^+ \mathbb{Z} + \mathbb{Z}[G_n]^-$  defined by  $f \mapsto \beta(f)$  gives rise to the exact sequence of  $\mathbb{Z}[G_n]$ -modules.*

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}[G_n]}(I_n, \mu_{2p^{n+1}}) \longrightarrow \mathcal{W}_n^- \longrightarrow (\text{Ann}_{\mathbb{Z}[G_n]} \text{Cl}(k_n))^- \longrightarrow B_n \longrightarrow 0$$

where  $B_n$  is a finite abelian elementary 2-group.

**Proof:**

By the definition of  $\mathcal{W}_n$  one has

$$\text{Hom}_{\mathbb{Z}[G_n]}(I_n, \mu_{2p^{n+1}}) = \{f \in \mathcal{W}_n \mid \beta(f) = 0\} = \ker \beta.$$

Note that  $f \in \mathcal{W}_n^-$  implies  $\beta(f) \in \mathbb{Z}[G_n]^-$ . Take  $f \in \mathcal{W}_n^-$  and  $\mathfrak{p}$  a prime ideal in  $I_n$ . Let  $\mathfrak{p}$  be a split prime ideal in  $I_n$ . Let  $m \geq 1$  be such that  $\mathfrak{p}^m$  is principal. Then

$$\mathfrak{p}^{m\beta(f)} = f(\mathfrak{p})^m \mathcal{O}_{k_n},$$

that implies

$$\mathfrak{p}^{\beta(f)} = f(\mathfrak{p}) \mathcal{O}_{k_n}.$$

Thus  $\beta(f) \in (\text{Ann}_{\mathbb{Z}[G_n]} \text{Cl}(k_n))^-$ .

Let  $\beta$  be in  $(\text{Ann}_{\mathbb{Z}[G_n]} \text{Cl}(k_n))^-$  and  $\mathfrak{p}$  a prime ideal in  $I_n$ . Then there exists  $\gamma \in k_n^*$  such that  $\mathfrak{p}^\beta = \gamma \mathcal{O}_{k_n}$ . Let  $\bar{\gamma}$  be the complex conjugate of  $\gamma$ . Then  $\bar{\gamma} = \gamma^{-1} \varepsilon$  for some  $\varepsilon \in \mathcal{O}_{k_n}^*$ . Thus  $\varepsilon = \gamma \bar{\gamma}$ , i.e.  $\varepsilon$  is a real unit. Consider  $\gamma_1 = \varepsilon^{-1} \gamma^2$ . One has:  $\gamma_1^{-1} = \bar{\gamma}_1$  and

$$\mathfrak{p}^{2\beta} = \gamma^2 \mathcal{O}_{k_n} = \varepsilon^{-1} \gamma^2 \mathcal{O}_{k_n} = \gamma_1 \mathcal{O}_{k_n}.$$

Let  $\gamma_2 \in k_n^*$  such that  $\mathfrak{p}^{2\beta} = \gamma_1 \mathcal{O}_{k_n} = \gamma_2 \mathcal{O}_{k_n}$  and  $\bar{\gamma}_2 = \gamma_2^{-1}$ . Then  $\gamma_1 = \gamma_2 \eta$  for some  $\eta \in \mathcal{O}_{k_n}^*$ ,  $\gamma_1^{-1} = \gamma_2^{-1} \bar{\eta}$ . That implies  $\eta \bar{\eta} = 1$ , i.e.  $\eta$  is a root of unity.

Now one can choose, for any  $\mathfrak{p} \in I_n$ ,  $\gamma_{\mathfrak{p}} \in k_n^*$  such that  $\mathfrak{p}^{2\beta} = \gamma_{\mathfrak{p}} \mathcal{O}_{k_n}$ ,  $\bar{\gamma}_{\mathfrak{p}} = \gamma_{\mathfrak{p}}^{-1}$  and  $\gamma_{\mathfrak{p}\sigma} = \gamma_{\mathfrak{p}}^\sigma \forall \sigma \in G_n$ . We set:

$$f(\mathfrak{p}) = \gamma_{\mathfrak{p}}$$

and one can verify that  $f \in \mathcal{W}_n^-$  and  $\beta(f) = 2\beta$ . Thus

$$2(\text{Ann}_{\mathbb{Z}[G_n]} \text{Cl}(k_n))^- \subset \beta(\mathcal{W}_n^-) \subset (\text{Ann}_{\mathbb{Z}[G_n]} \text{Cl}(k_n))^-,$$

that completes the proof.  $\square$

Let  $l \neq p$  be a prime number. Let  $\mathfrak{l}$  be the prime ideal of  $k_n$  above  $l$  and  $q = |\mathcal{O}_{k_n}/\mathfrak{l}|$ . Fix a primitive  $l$ -th root of unity  $\zeta_l$ . The Gauss sum  $\tau_n(\mathfrak{l})$  associated to  $\mathfrak{l}$  is defined by

$$\tau_n(\mathfrak{l}) = - \sum_{a \in \mathbb{F}_q} \chi_{\mathfrak{l}}(a) \zeta_l^{\text{Tr}_{\mathbb{F}_q/\mathbb{F}_l}(a)}$$

where  $\chi_{\mathfrak{l}}$  is a character on  $\mathbb{F}_q^*$  of order  $p^{n+1}$  defined by

$$\chi_{\mathfrak{l}}(x) \equiv x^{-\frac{q-1}{p^{n+1}}} \pmod{\mathfrak{l}}.$$

One can show that  $\forall \delta \in G_n$  one has  $\tau_n(\mathfrak{l}^\delta) = \tau_n(\mathfrak{l})^\delta$  (see [W, §6.1]. So we have a well defined morphism of  $\mathbb{Z}[G_n]$ -modules

$$\tau_n : I_n \longrightarrow \Omega(\zeta_{p^{n+1}})^*,$$

where  $\Omega$  is the compositum of all the  $\mathbb{Q}(\zeta_m)$ ,  $m$  prime to  $p$ .

**Definition 3** The Jacobi module  $\mathcal{J}_n$  associated to  $k_n$  is defined by

$$\mathcal{J}_n = \mathbb{Z}[G_n]\tau_n \cap \text{Hom}_{\mathbb{Z}[G_n]}(I_n, k_n^*),$$

and the module of Jacobi sums  $J_n$  is defined by

$$J_n = \{f(\mathfrak{a}) \mid f \in \mathcal{J}_n, \mathfrak{a} \in I_n\}.$$

Let us denote by  $\sigma_a$  the image of  $a \in \mathbb{Z}$ , prime to  $p$ , via the standard isomorphism  $(\mathbb{Z}/p^{n+1}\mathbb{Z})^* \simeq G_n$ . Let

$$\theta_n = \frac{1}{p^{n+1}} \sum_{a=1, (a,p)=1}^{p^{n+1}} a\sigma_a^{-1}$$

be the Stickelberger element of  $k_n$ . Set

$$\mathcal{S}'_n = \sum_{(t,p)=1} \mathbb{Z}[G_n](t - \sigma_t).$$

**Definition 4** The Stickelberger ideal of  $k_n$  is defined by

$$\mathcal{S}_n = \mathcal{S}'_n \theta_n,$$

(see [W, Lemma 6.9]).

**Theorem 1 (Stickelberger's Theorem [S, Theorem3.1])** Let  $\mathfrak{p}$  be a prime ideal in  $I_n$ , and  $\beta \in \mathcal{S}'_n$ . Then  $\tau(\mathfrak{p})^\beta \in k_n^*$ ,  $\beta\theta_n \in \mathbb{Z}[G_n]$  and

$$\tau_n(\mathfrak{p})^\beta \mathcal{O}_{k_n} = \mathfrak{p}^{\beta\theta_n}.$$

Moreover,  $\tau_n^\beta(\mathfrak{p}) \in \mathcal{U}_n$ .

**Lemma 2**

$$\mathcal{S}'_n = \{\beta \in \mathbb{Z}[G_n] \mid \tau_n^\beta \in \mathcal{J}_n\}.$$

**Proof:** The inclusion  $\mathcal{S}'_n \subset \{\beta \in \mathbb{Z}[G_n] \mid \tau_n^\beta \in \mathcal{J}_n\}$  is obvious by Stickelberger's theorem.

To prove the inverse inclusion it suffices to show that  $\tau_n^\beta \in \mathcal{J}_n$  implies  $\beta\theta_n \in \mathbb{Z}[G_n]$ .

Let  $\mathfrak{p} \in I_n$  be a split prime ideal and  $\tilde{\mathfrak{p}}$  the unique prime ideal of  $\mathbb{Z}[\zeta_{p^{n+1}}, \zeta_l]$  above  $\mathfrak{p}$ , where  $l = \mathfrak{p} \cap \mathbb{Q}$ ,  $l \equiv 1 \pmod{p^{n+1}}$ . Then

$$\tau_n(\mathfrak{p})\mathbb{Z}[\zeta_{p^{n+1}}, \zeta_l] = \tilde{\mathfrak{p}}^{(l-1)\theta_n}.$$

Thus

$$\tau_n(\mathfrak{p})^\beta \mathbb{Z}[\zeta_{p^{n+1}}, \zeta_l] = \tilde{\mathfrak{p}}^{(l-1)\beta\theta_n}.$$

On the other hand,

$$\tau_n(\mathfrak{p})^\beta \mathcal{O}_{k_n} = \mathfrak{p}^z$$

for some  $z \in \mathbb{Z}[G_n]$ , that implies

$$\tilde{\mathfrak{p}}^{(l-1)z} = \tilde{\mathfrak{p}}^{(l-1)\theta_n\beta}$$

and since  $l \equiv 1 \pmod{p^{n+1}}$ , one has  $(l-1)z = (l-1)\theta_n\beta$ . Thus  $z = \theta_n\beta$  that implies  $\beta\theta_n \in \mathbb{Z}[G_n]$ .  $\square$

**Proposition 2**

$$(1) \mathcal{J}_n \subset \mathcal{W}_n$$

$$(2) \mathcal{J}_n \simeq \mathcal{S}_n.$$

**Proof:**

(1) Using the lemma 2 one can easily verify that

$$\mathcal{J}_n = \{\tau_n^\delta \mid \delta \in \mathcal{S}'_n\}.$$

Then for any  $f \in \mathcal{J}_n$  there exists  $\delta \in \mathcal{S}'_n$  such that  $f = \tau_n^\delta$ .

Let  $f \in \mathcal{J}_n$  and let  $\mathfrak{a} \in I_n$  be a principal ideal,  $\mathfrak{a} = \alpha \mathcal{O}_{k_n}$ . Then by the Stickelberger Theorem one has

$$f(\mathfrak{a}) = \tau_n^\delta(\mathfrak{a}) = \varepsilon \alpha^{\delta \theta_n}$$

for some unit  $\varepsilon$ . But

$$\tau_n(\mathfrak{a}) \overline{\tau_n(\mathfrak{a})} = N_n(\mathfrak{a}) = N_n(\alpha),$$

so  $\varepsilon \in \mu_{2p^{n+1}}$ . That means  $f(\mathfrak{a}) \equiv \alpha^{\delta \theta_n} \pmod{\mu_{2p^{n+1}}}$ , i.e.  $f \in \mathcal{W}_n$ .

(2) As  $\mathcal{J}_n \subset \mathcal{W}_n$  by (1), the map  $\beta|_{\mathcal{J}_n}$  is well defined. On the other hand, for any  $f \in \mathcal{J}_n$

$$\beta(f) = \beta(\tau_n^\delta) = \delta \theta_n$$

for some  $\delta \in \mathcal{S}'_n$ . Thus one has a well defined map

$$\begin{array}{ccc} \mathcal{J}_n & \longrightarrow & \mathcal{S}_n \\ \tau_n^\delta & \longmapsto & \delta \theta_n \end{array}$$

This map is obviously surjective (by the definition of  $\mathcal{S}_n$ ). Its kernel is a submodule of  $\text{Hom}_{\mathbb{Z}[G_n]}(I_n, \mu_{p^{n+1}})$  by proposition 1. Let  $\delta \in \mathbb{Z}[G_n]$  such that  $\delta \theta_n = 0$ . Then we have  $\sigma_{-1} \delta = \delta$  ( $\sigma_{-1}$  being the complex conjugation in  $G_n$ ) and  $\delta N_n = 0$ .

Now let  $\delta \in \mathcal{S}'_n$  such that  $\delta \theta_n = 0$ . Then

$$\tau_n^{\sigma_{-1} \delta} = \tau_n^\delta,$$

and

$$\tau_n^{\delta \sigma_{-1}} = (\tau_n^{\sigma_{-1}})^\delta = \tau_n^{-\delta}$$

as  $\theta_n \theta_n^{\sigma_{-1}} = N_n$ . Thus  $\tau_n^{2\delta} = 1$ . Therefore  $\tau_n^\delta = 1$  as  $\tau_n^\delta \equiv 1 \pmod{\pi_n}$ .  $\square$

**Lemma 3** *Let  $N_{n,n-1}$  be the norm map in the extension  $k_n/k_{n-1}$  and  $\mathfrak{L} \in I_n$  a prime ideal. Then*

$$N_{n,n-1}(\tau_n(\mathfrak{L})) = \tau_{n-1}(N_{n,n-1}(\mathfrak{L})) \zeta^a l^b,$$

for some  $a, b \in \mathbb{Z}$  and some  $\zeta \in \mu_{p^{n+1}}$ .

For a proof see [I1, Lemma 2].

**Remark 1** *The composition  $N_{n,n-1} \circ \tau_n$  is well defined because  $\text{Gal}(k_n/k_{n-1}) \simeq \text{Gal}(\Omega(\zeta_{p^{n+1}})/\Omega(\zeta_{p^n}))$ ,  $\forall \geq 1$ .*

**Lemma 4** ([W, Proposition 7.6 (c)]) *The restriction map  $\text{Res} : \mathbb{Z}[G_n] \rightarrow \mathbb{Z}[G_{n-1}]$  induces the surjective map*

$$\text{Res} : \mathcal{S}_n^- \longrightarrow \mathcal{S}_{n-1}^-.$$

**Proposition 3**

$$\forall n \geq 1, N_{n,n-1} \circ \mathcal{J}_n^- \equiv J_{n-1}^- \circ N_{n,n-1} \pmod{\text{Hom}_{\mathbb{Z}[G_n]}(I_n, \mu_{2p^{n+1}})}$$

**Proof:** Let  $f \in \mathcal{J}_n^-$ . By the proposition 2 there exists some  $\beta \in \mathcal{S}_n^-$  such that  $f = \tau_n^\beta$ . Let  $\mathfrak{L} \in I_n$  be a prime ideal and  $\mathfrak{l} = N_{n,n-1}$ . Then by the lemmas 3 and 4

$$N_{n,n-1}(\tau_n^\beta(\mathfrak{L})) \equiv \tau_{n-1}^{\text{Res}(\beta)}(\mathfrak{l}) \pmod{\mu_{p^{n+1}}(\mathfrak{L})}.$$

The Proposition follows.  $\square$

### 3 Annihilators

We recall that  $\psi$  is an odd  $\mathbb{Q}_p$ -valued character of  $\Delta$ , irreducible over  $\mathbb{Q}_p$ , different from Teichmüller character  $\omega$ .

**Lemma 5** *Let  $\mathcal{M} \in \Lambda$  be the minimal polynomial of  $e_\psi X$ . Then*

$$\varprojlim e_\psi(\text{Ann}_{\mathbb{Z}_p[G_n]} A_n) = \mathcal{M}(T)\Lambda,$$

*the projective limit being taken for the restriction maps.*

**Proof:** First we remark that

$$e_\psi \text{Ann}_{\mathbb{Z}_p[G_n]} A_n = \text{Ann}_{\mathbb{Z}_p[\Gamma_n]} e_\psi A_n.$$

We set  $A_{n,\psi} = e_\psi A_n$  for simplicity.

Let  $\mathcal{M} = (\mathcal{M}_n)_{n \geq 0} \in \Lambda \simeq \varprojlim \mathbb{Z}_p[\Gamma_n]$ , the limit being taken with respect for restriction maps. As  $X^-$  has no nontrivial finite submodule (see [W, Proposition 13.28]),  $\mathcal{M}_n$  annihilates  $A_{n,\psi}$ , that means  $\mathcal{M}_n \mathbb{Z}_p[\Gamma_n] \subset \text{Ann}_{\mathbb{Z}_p[\Gamma]} A_{n,\psi}$ . Thus

$$\mathcal{M}(T)\Lambda \subset \varprojlim \text{Ann}_{\mathbb{Z}_p[\Gamma]} A_{n,\psi}.$$

Let  $\delta = (\delta_n)_{(n \geq 0)} \in \varprojlim \text{Ann}_{\mathbb{Z}_p[\Gamma]} A_{n,\psi}$ . Then for any  $n \geq 0$   $\delta_n A_{n,\psi} = \{0\}$ . On the other hand,

$$e_\psi X = \varprojlim A_{n,\psi}.$$

Then  $\delta e_\psi X = \{0\}$ , that implies

$$\delta \in \text{Ann}_\Lambda e_\psi X = \mathcal{M}(T)\Lambda,$$

that completes the proof.  $\square$

Let  $\overline{\mathcal{W}}_n = \mathcal{W}_n \otimes \mathbb{Z}_p$  the  $p$ -adic adherence of  $\mathcal{W}_n$ . The map  $\beta$  of Proposition 1 induces the map

$$\begin{aligned} \overline{\mathcal{W}}_n^- &\longrightarrow (\text{Ann}_{\mathbb{Z}[G_n]} Cl(k_n))^- \otimes_{\mathbb{Z}} \mathbb{Z}_p = (\text{Ann}_{\mathbb{Z}_p[G_n]} A_n)^- \\ w \otimes a &\longmapsto a\beta(w) \end{aligned}$$

that we shall always note  $\beta$ . Thus we have the short exact sequence

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}[G_n]}(I_n, \mu_{2p^{n+1}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow \overline{\mathcal{W}}_n^- \longrightarrow (\text{Ann}_{\mathbb{Z}_p[G_n]} A_n)^- \longrightarrow 0.$$

Applying  $e_\psi$  to all the terms of this sequence we get an isomorphism of  $\mathbb{Z}_p[G_n]$ -modules

$$e_\psi \overline{\mathcal{W}}_n \simeq e_\psi \text{Ann}_{\mathbb{Z}_p[G_n]} A_n = \text{Ann}_{\mathbb{Z}_p[\Gamma_n]} A_{n,\psi}. \quad (2)$$

Let  $z \in \overline{\mathcal{W}}_{n,\psi} = e_\psi \overline{\mathcal{W}}_n$ . Then  $z$  induces naturally by class field theory (see [Iw, p.455]) a morphism of  $\mathbb{Z}_p[\Gamma_n]$ -modules:

$$z : \mathfrak{X}_{n,\psi} \longrightarrow \mathcal{U}_{n,\psi}. \quad (3)$$

**Lemma 6** *Let  $z \in \overline{\mathcal{W}}_{n,\psi}$  such that  $\beta(z) \in \mathbb{Q}_p[\Gamma_n]^*$ . Then the kernel of  $z$  is  $\text{Tor}_{\mathbb{Z}_p[\Gamma_n]} \mathfrak{X}_{n,\psi}$ .*

**Proof:** We have  $z(\mathcal{U}_{n,\psi}) = \mathcal{U}_{n,\psi}^{\beta(z)}$ . As  $\beta \in \mathbb{Q}_p[\Gamma_n]^*$ , the quotient  $\mathcal{U}_{n,\psi}/\mathcal{U}_{n,\psi}^{\beta(z)}$  is finite. Thus

$$\text{rank}_{\mathbb{Z}_p} z(\mathfrak{X}_{n,\psi}) = \text{rank}_{\mathbb{Z}_p} \mathfrak{X}_{n,\psi} = \text{rank}_{\mathbb{Z}_p} \mathcal{U}_{n,\psi}.$$

Thus  $\ker(z : \mathfrak{X}_{n,\psi} \longrightarrow \mathcal{U}_{n,\psi}) = \text{Tor}_{\mathbb{Z}_p[\Gamma_n]} \mathfrak{X}_{n,\psi}$ .  $\square$

For any  $n \in \mathbb{N}$  let  $\mathcal{M}_n \in \mathbb{Z}_p[G_n]$  such that

$$\mathcal{M}_n \equiv \mathcal{M} \pmod{\omega_n}.$$

Then  $(\mathcal{M}_n)_{n \geq 0} = \mathcal{M}$  in  $\Lambda$ . Let  $w_n \in \overline{\mathcal{W}}_{n,\psi} = e_\psi \overline{\mathcal{W}}_n$  be the element of  $\overline{\mathcal{W}}_{n,\psi}$  corresponding to  $\mathcal{M}_n$  via the homomorphism (2).

**Remark 2** *The Lemma 6 is applicable to  $w_n$ , and to the map that consists in multiplication by  $e_\psi \theta_n$ .*

**Lemma 7** *Let  $\overline{J}_n = J_n \otimes_{\mathbb{Z}} \mathbb{Z}_p$  be the  $p$ -adic adherence of  $J_n$  in  $\mathcal{U}_n$  and  $\overline{W}_n = W_n \otimes_{\mathbb{Z}} \mathbb{Z}_p$  the  $p$ -adic adherence of  $W_n$  in  $\mathcal{U}_n$ . Then*

$$e_\psi \overline{J}_n \subset w_n(\mathfrak{X}_{n,\psi}) \subset e_\psi \overline{W}_n.$$

**Proof :** One can verify that  $\beta(e_\psi \overline{J}_n) = e_\psi \theta_n \mathbb{Z}_p[\Gamma_n]$  (see [W, Chap. 7]). By the Main Conjecture (see [W, §13.6])

$$e_\psi \theta_n \mathbb{Z}_p[\Gamma_n] \subset \mathcal{M}_n \mathbb{Z}_p[\Gamma_n].$$

Set  $\widetilde{\mathcal{W}}_{n,\psi}$  the sub- $\mathbb{Z}_p[\Gamma_n]$ -module of  $\overline{\mathcal{W}}_{n,\psi}$  generated by  $w_n$ . Then  $\beta(\widetilde{\mathcal{W}}_{n,\psi}) = \mathcal{M}_n \mathbb{Z}_p[\Gamma_n]$ . Thus

$$\beta(e_\psi \overline{J}_n) \subset \beta(\widetilde{\mathcal{W}}_{n,\psi}).$$

As  $\beta$  is an isomorphism, this is equivalent to

$$e_\psi \overline{J}_n \subset \widetilde{\mathcal{W}}_{n,\psi},$$

That implies

$$e_\psi \overline{J}_n \subset w_n(\mathfrak{X}_{n,\psi}). \quad \square$$

Take  $(z_n)_{n \geq 1}$ ,  $z_n \in \widetilde{W}_{n,\psi}$  such that  $\forall n \geq 1$ ,  $\text{Res}_{n+1,n}\beta(z_{n+1}) = \beta(z_n)$ . By the class field theory the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{X}_{n+1,\psi} & \xrightarrow{z_{n+1}} & \mathcal{U}_{n+1,\psi} \\ \downarrow \text{Res}_{n+1,n} & & \downarrow N_{n+1,n} \\ \mathfrak{X}_{n,\psi} & \xrightarrow{z_n} & \mathcal{U}_{n,\psi} \end{array}$$

so the map

$$z_\infty : \mathfrak{X}_{\infty,\psi} \longrightarrow \mathcal{U}_{\infty,\psi} \quad (4)$$

is naturally well defined and

$$z_\infty(\mathfrak{X}_{\infty,\psi}) = \varprojlim z_n(\mathfrak{X}_{n,\psi}) \subseteq \mathcal{U}_{\infty,\psi}$$

**Lemma 8** *The kernel of  $z_\infty$  is isomorphic to  $\alpha(e_{\omega\psi^{-1}}X)$ , where  $\alpha(e_{\omega\psi^{-1}}X)$  is the Iwasawa adjoint module of  $e_{\omega\psi^{-1}}X$ .*

**Proof :** By the definition of  $z_\infty$ ,  $\ker z_\infty = \varprojlim \ker z_n = \varprojlim \text{Tor}_{\mathbb{Z}_p[\Gamma_n]} \mathfrak{X}_{n,\psi}$ . But  $\varprojlim \text{Tor}_{\mathbb{Z}_p[\Gamma_n]} \mathfrak{X}_{n,\psi} \simeq \alpha(e_{\omega\psi^{-1}}X)$  (see [N1, Proposition 3.1]).  $\square$

Take  $z_n = w_n \forall n \geq 1$ . Then  $e_\psi \overline{J}_\infty \subset w_\infty(\mathfrak{X}_{\infty,\psi})$  by the Lemma 7.

**Lemma 9**

$$w_\infty(e_\psi \mathcal{U}_\infty) = e_\psi \mathcal{M} \mathcal{U}_\infty$$

**Proof :** Obvious as  $e_\psi \mathcal{U}_\infty$  is free of rank 1.  $\square$

**Lemma 10** *The module  $W_{\infty,\psi} = \varprojlim \overline{W}_{n,\psi}$  is pseudo-isomorphic to  $w_\infty(\mathfrak{X}_{\infty,\psi})$ .*

**Proof:** Let  $E$  be the elementary  $\Lambda$ -module such that

$$0 \longrightarrow e_\psi X \longrightarrow E \longrightarrow B \longrightarrow 0,$$

where  $B$  is a finite  $\Lambda$ -module. Then  $\forall n \gg 0$ ,  $\omega_n B = \{0\}$ , and by the snake lemma we obtain the exact sequence

$$0 \longrightarrow B \longrightarrow e_\psi A_n \longrightarrow E/\omega_n E \longrightarrow B \longrightarrow 0.$$

Let  $Y_n = \text{Ann}_{\mathbb{Z}_p[\Gamma_n]} E/\omega_n E = \mathcal{M}_n \mathbb{Z}_p[\Gamma_n]$ . It is a submodule of  $Z_n = \text{Ann}_{\mathbb{Z}_p[\Gamma_n]} e_\psi A_n \simeq e_\psi \overline{W}_n$ , so there exists a submodule  $\widehat{W}_n$  of  $e_\psi \overline{W}_n$  such that  $\widehat{W}_n \simeq Y_n$  as  $\mathbb{Z}_p[\Gamma_n]$ -modules.  $Y_n$  being monogenous, the same is for  $\widehat{W}_n$ , so it is generated by  $w_n$ . Thus  $\widehat{W}_n = \widetilde{W}_{n,\psi}$ .

There exists  $\delta \in \Lambda$ , prime to  $\mathcal{M}$ , such that  $\delta B = \{0\}$ . Then  $\delta Z_n \subset Y_n$ , i.e.  $\delta e_\psi \overline{W}_n \subset \widetilde{W}_{n,\psi}$ . In particular that means

$$\delta e_\psi \overline{W}_n \subset \widetilde{W}_{n,\psi} \subset e_\psi \overline{W}_n, \quad (5)$$

where  $\widetilde{W}_n = w_n(\mathfrak{X}_{n,\psi})$ . So, taking the projective limit in (5) we obtain

$$\delta e_\psi \overline{W}_\infty \subset w_\infty(e_\psi \mathfrak{X}_\infty) \subset e_\psi \overline{W}_\infty.$$

Thus the quotient module  $e_\psi \overline{W}_\infty / w_\infty(e_\psi \mathfrak{X}_\infty)$  is annihilated by two relatively prime polynomials  $\delta$  and  $\mathcal{M}$ , i.e. is finite (see [W, §13.2]).  $\square$

The classical class field theory sequence

$$0 \longrightarrow \overline{\mathcal{O}_{k_n}^* \cap \mathcal{U}_n} \longrightarrow \mathcal{U}_n \longrightarrow \mathfrak{X}_n \longrightarrow A_n \longrightarrow 0$$

gives by taking the  $\psi$ -parts the short exact sequence

$$0 \longrightarrow e_\psi \mathcal{U}_n \longrightarrow e_\psi \mathfrak{X}_n \longrightarrow e_\psi A_n \longrightarrow 0, \quad (6)$$

as  $\psi \neq \omega$ .

Passing to the projective limit in this sequence we obtain the short exact sequence

$$0 \longrightarrow e_\psi \mathcal{U}_\infty \longrightarrow e_\psi \mathfrak{X}_\infty \longrightarrow e_\psi X \longrightarrow 0. \quad (7)$$

## Theorem 2

$$\text{char}_\Lambda(e_\psi \overline{W}_\infty / e_\psi \overline{J}_\infty) = (\text{char}_\Lambda e_\psi X_\infty) / \mathcal{M}(T).$$

**Proof:** By the Lemma 9, the map  $w_\infty$  gives rise to the map

$$\overline{w}_\infty : \frac{e_\psi \mathfrak{X}_\infty}{e_\psi \mathcal{U}_\infty} \longrightarrow \frac{e_\psi \mathcal{U}_\infty}{\mathcal{M} e_\psi \mathcal{U}_\infty}.$$

So in virtue of the sequence (7), we have the map

$$\overline{w}_\infty : e_\psi X \longrightarrow \frac{e_\psi \mathcal{U}_\infty}{\mathcal{M} e_\psi \mathcal{U}_\infty}.$$

The kernel of the map  $w_\infty$  being the  $\Lambda$ -torsion module isomorphic to  $\alpha(e_{\omega\psi^{-1}} X)$  and  $e_\psi \mathcal{U}_\infty$  being a free  $\Lambda$ -module,  $\ker(w_\infty) \cap e_\psi \mathcal{U}_\infty = \{0\}$ . So  $\ker(\overline{w}_\infty) \simeq \alpha(e_{\omega\psi^{-1}} X)$ . And we have the following exact sequence

$$0 \longrightarrow \alpha(e_{\omega\psi^{-1}} X) \longrightarrow e_\psi X \longrightarrow e_\psi \mathcal{U}_\infty / \mathcal{M} e_\psi \mathcal{U}_\infty.$$

Let  $F = \text{char}_\Lambda e_\psi X$  the characteristic polynomial of  $e_\psi X$  and  $\theta_{\infty,\psi} = (e_\psi \theta_n)_{n \geq 0}$ . Then by the Main Conjecture and by the Lemma 7 we obtain the second exact sequence

$$0 \longrightarrow \alpha(e_{\omega\psi^{-1}} X) \longrightarrow e_\psi X \xrightarrow{\times \theta_{\infty,\psi}} e_\psi \mathcal{U}_\infty / F e_\psi \mathcal{U}_\infty$$

These two sequences give two short exact sequences

$$0 \longrightarrow \alpha(e_{\omega\psi^{-1}} X) \longrightarrow e_\psi X \longrightarrow w_\infty(\mathfrak{X}_{\infty,\psi}) / \mathcal{M} e_\psi \mathcal{U}_\infty \longrightarrow 0$$

and

$$0 \longrightarrow \alpha(e_{\omega\psi^{-1}} X) \longrightarrow e_\psi X \longrightarrow e_\psi \overline{J}_\infty / F e_\psi \mathcal{U}_\infty \longrightarrow 0,$$

as  $\theta_\infty \mathfrak{X}_{\infty, \psi} = e_\psi \overline{\mathcal{J}}_\infty$  (see [B, Lemme 8]). Thus

$$\text{char}_\Lambda w_\infty(\mathfrak{X}_{\infty, \psi}) / \mathcal{M}e_\psi \mathcal{U}_\infty = \text{char}_\Lambda e_\psi \overline{\mathcal{J}}_\infty / Fe_\psi \mathcal{U}_\infty. \quad (8)$$

Set  $e_\psi \widetilde{W}_\infty = w_\infty(\mathfrak{X}_{\infty, \psi})$ . The tautological short exact sequence

$$0 \longrightarrow e_\psi \overline{\mathcal{J}}_\infty \longrightarrow e_\psi \widetilde{W}_\infty \longrightarrow e_\psi \widetilde{W}_\infty / e_\psi \overline{\mathcal{J}}_\infty \longrightarrow 0$$

gives rise to the short exact sequence

$$0 \longrightarrow \frac{e_\psi \overline{\mathcal{J}}_\infty}{F\mathcal{U}_{\infty, \psi}} \longrightarrow \frac{e_\psi \widetilde{W}_\infty}{F\mathcal{U}_{\infty, \psi}} \longrightarrow \frac{e_\psi \widetilde{W}_\infty}{e_\psi \overline{\mathcal{J}}_\infty} \longrightarrow 0.$$

Thus

$$\text{char} \frac{e_\psi \overline{\mathcal{J}}_\infty}{F\mathcal{U}_{\infty, \psi}} = \text{char} \frac{e_\psi \widetilde{W}_\infty}{F\mathcal{U}_{\infty, \psi}} \left( \text{char} \frac{e_\psi \widetilde{W}_\infty}{e_\psi \overline{\mathcal{J}}_\infty} \right)^{-1}. \quad (9)$$

In the same way, the sequence

$$0 \longrightarrow \mathcal{M}e_\psi \mathcal{U}_\infty \longrightarrow e_\psi \widetilde{W}_\infty \longrightarrow e_\psi \widetilde{W}_\infty / \mathcal{M}e_\psi \mathcal{U}_\infty \longrightarrow 0$$

gives rise to the sequence

$$0 \longrightarrow \frac{\mathcal{M}\mathcal{U}_{\infty, \psi}}{F\mathcal{U}_{\infty, \psi}} \longrightarrow \frac{e_\psi \widetilde{W}_\infty}{F\mathcal{U}_{\infty, \psi}} \longrightarrow \frac{e_\psi \widetilde{W}_\infty}{\mathcal{M}\mathcal{U}_{\infty, \psi}} \longrightarrow 0$$

Thus

$$\text{char} \frac{e_\psi \widetilde{W}_\infty}{\mathcal{M}\mathcal{U}_{\infty, \psi}} = \text{char} \frac{e_\psi \widetilde{W}_\infty}{F\mathcal{U}_{\infty, \psi}} \left( \text{char} \frac{\mathcal{M}\mathcal{U}_{\infty, \psi}}{F\mathcal{U}_{\infty, \psi}} \right)^{-1}. \quad (10)$$

Comparing the equalities (8), (9) and (10) we obtain the equality

$$\text{char} \frac{\mathcal{M}\mathcal{U}_{\infty, \psi}}{F\mathcal{U}_{\infty, \psi}} = \text{char} \frac{e_\psi \widetilde{W}_\infty}{e_\psi \overline{\mathcal{J}}_\infty}.$$

In virtue of the Lemma 10,

$$\text{char} \frac{e_\psi \widetilde{W}_\infty}{e_\psi \overline{\mathcal{J}}_\infty} = \text{char} \frac{e_\psi \overline{W}_\infty}{e_\psi \overline{\mathcal{J}}_\infty}.$$

As  $\mathcal{U}_{\infty, \psi}$  is free of rank 1,

$$\text{char} \frac{\mathcal{M}\mathcal{U}_{\infty, \psi}}{F\mathcal{U}_{\infty, \psi}} = \frac{F(T)}{\mathcal{M}(T)}.$$

So

$$\text{char} \frac{e_\psi \overline{W}_\infty}{e_\psi \overline{\mathcal{J}}_\infty} = \frac{F(T)}{\mathcal{M}(T)}. \quad \square$$

**Corollary 1** (cf. [B, Théorème 1])

$$\text{char}_\Lambda \frac{\mathcal{U}_{\infty, \psi}}{e_\psi \overline{\mathcal{J}}_\infty} = \text{char}_\Lambda \alpha(e_{\omega\psi^{-1}} X).$$

**Corollary 2** *The module  $e_\psi X$  is pseudo-monogenous if and only if the quotient module  $e_\psi \overline{W}_\infty / e_\psi \overline{J}_\infty$  is finite.*

By the corollary 1, we see that Greenberg Conjecture implies that

$$\frac{e_\psi \mathcal{U}_\infty}{e_\psi \overline{W}_\infty} \text{ is finite.}$$

So it is natural to ask the following question.

**Question:** Let  $p$  be an odd prime number. Let  $\psi$  be an odd character of  $\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ ,  $\psi \neq \omega$ , then, is it true that

$$\frac{e_\psi \mathcal{U}_\infty}{e_\psi \overline{W}_\infty} \text{ is finite ?}$$

**Remark 3** *Note that the positive answer to this question is equivalent to*

$$\text{char } e_\psi X = \text{char } \alpha(e_{\omega\psi^{-1}} X) \times \mathcal{M},$$

*so it implies weak Greenberg Conjecture (see [BN], [N2]).*

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Tatiana BELIAEVA, Bruno ANGLES  
Université de Caen,  
Laboratoire Nicolas Oresme, CNRS 6139,  
Campus II, Boulevard Maréchal Juin,  
B.P. 5186, 14032 Caen Cedex,, France.

bruno.angles@math.unicaen.fr  
tatiana.believa@math.unicaen.fr